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The Yangian Deformation of the W-Algebras and the Calogero-Sutherland System. *

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ABSTRACT

The Yangian symmetry Y(su(n)) of the Calogero-Sutherland-Moser spin model is reconsidered. The Yangian generators are constructed from two non-commuting su(n)-loop algebras. The latters generate an infinite dimensional symmetry algebra which is a deformation of the W_∞-algebra. We show that this deformed W_∞-algebra contains an infinite number of Yangian subalgebras with different deformation parameters.

Introduction.

We consider the Calogero and the Sutherland spin models. These models are inverse square interacting N-body systems with internal degree of freedom. Their hamiltonians are respectively defined as

$$\mathcal{H}_C = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j\neq k}^{N} \frac{\lambda^2 - \lambda\, P_{jk}}{(x_j - x_k)^2}, \tag{1}$$

$$\mathcal{H}_S = \sum_{j=1}^{N} \Big(x_j \frac{\partial}{\partial x_j} \Big)^2 + \sum_{j\neq k}^{N} (-\lambda^2 + \lambda\, P_{jk}) \frac{x_j\, x_k}{(x_j - x_k)^2}. \tag{2}$$

Here P_{jk} is the permutation operator in the su(n) spin space. Using the spin operators E^{ab} as basis of the su(n) algebra, $E^{ab} \equiv |a\rangle\langle b|$ $(a, b = 1, \cdots, n)$, the operator P_{jk} can

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be written as

$$P_{jk} = \sum_{a,b=1}^{n} E_j^{ab} E_k^{ba}. \tag{3}$$

The hamiltonian (2) reduces to a dynamical particle system with periodic boundary condition when we change the variables, $x_j \to \exp(-2\pi \mathrm{i} z_j/L)$.

Both the Calogero (1) and the Sutherland (2) models are integrable.[1–5] The integrability can be checked by several methods: e.g. the quantum Lax formalism and the Dunkl operator formalism. Usually, the Calogero and the Sutherland spin models are treated separately. In this note we construct an algebra in which both models are naturally included.[1] Our algebra can be generated from two non-commuting su(n) loop subalgebras. It has an infinite number of Yangian subalgebras.

The Calogero Model.

Let us first consider the Calogero system. For our purpose we define the following two operators:

$$J_0^{ab} = \sum_{j=1}^{N} E_j^{ab}, \tag{4}$$

$$J_1^{ab} = \sum_j E_j^{ab} \frac{\partial}{\partial x_j} - \lambda \sum_{j \neq k} (E_j E_k)^{ab} \frac{1}{x_j - x_k}. \tag{5}$$

Here we have used the conventional notations, $(E_j E_k)^{ab} = \sum_{c=1}^{n} E_j^{ac} E_k^{cb}$. The generators J_0^{ab} and J_1^{ab} satisfy the following relations:

$$[J_0^{ab}, J_0^{cd}] = \delta^{bc} J_0^{ad} - \delta^{da} J_0^{cb}, \tag{6}$$

$$[J_0^{ab}, J_1^{cd}] = \delta^{bc} J_1^{ad} - \delta^{da} J_1^{cb}, \tag{7}$$

$$[J_0^{ab}, [J_1^{cd}, J_1^{ef}]] - [J_1^{ab}, [J_0^{cd}, J_1^{ef}]] = 0. \tag{8}$$

The third equation is known as the Serre relation for the loop algebra. These relations imply that the higher generators $J_{n>1}^{ab}$, which are defined recursively using the generator J_1^{ab}, form a representation of su(n) loop algebra,

$$[J_n^{ab}, J_m^{cd}] = \delta^{bc} J_{n+m}^{ad} - \delta^{da} J_{n+m}^{cb}. \tag{9}$$

Remark that the generators of the su(n) loop algebra J_n^{ab} are conserved operators for the Calogero spin model,[5,6]

$$[\mathcal{H}_C, J_n^{ab}] = 0. \tag{10}$$

From this we conclude that the Calogero spin model (1) is su(n) loop invariant.

The Sutherland Model.

Consider now the Sutherland spin model (2). It can be viewed as the Calogero spin model with periodic boundary condition. It is not invariant under the su(n) loop algebra. However, the Sutherland spin model is invariant under a "deformed" su(n)

loop algebra, or in a recent mathematical terminology, a Yangian algebra Y(su(n)).[5,7] The Yangian algebra was first defined by Drinfeld as a Hopf algebra accompanied with the Yang's rational solution of the quantum Yang-Baxter equation.[8] To see that the Sutherland spin model (2) possesses the Yangian symmetry, we introduce two generators as,[5,7]

$$\begin{aligned} Q_0^{ab} &= J_0^{ab}, & (11)\\ Q_1^{ab} &= \sum_j E_j^{ab}\Big(x_j \frac{\partial}{\partial x_j} + \frac{1}{2}\Big) - \frac{\lambda}{2}\sum_{j\neq k}(E_j E_k)^{ab}\frac{x_j + x_k}{x_j - x_k}. & (12)\end{aligned}$$

One then directly check that generators Q_0^{ab} and Q_1^{ab} are conserved operators for the Sutherland spin model:

$$[Q_0^{ab}, \mathcal{H}_S] = [Q_1^{ab}, \mathcal{H}_S] = 0. \tag{13}$$

After a lengthy calculation we have the following commutation relations:

$$[J_0^{ab}, Q_1^{cd}] = \delta^{bc}\, Q_1^{ad} - \delta^{da}\, Q_1^{cb}, \tag{14}$$

$$\begin{aligned} &[J_0^{ab}, [Q_1^{cd}, Q_1^{ef}]] - [Q_1^{ab}, [J_0^{cd}, Q_1^{ef}]] \\ &= \frac{\lambda^2}{4}\Big([J_0^{ab}, [(J_0 J_0)^{cd}, (J_0 J_0)^{ef}]] - [(J_0 J_0)^{ab}, [J_0^{cd}, (J_0 J_0)^{ef}]]\Big). \end{aligned} \tag{15}$$

These relations together with equation (6) are the defining relations of the Yangian Y(su(n)). The second equation (15) is called the "deformed" Serre relation. It reduces to the Serre relation (8) for the loop algebra when $\lambda \to 0$. In this sense, the Yangian can be viewed as a "deformed" loop algebra. The relations (6) and (14-15) show that the generators Q_0^{ab} and Q_1^{ab} form a representation of the Yangian algebra Y(su(n)). Since the Yangian generators Q_n^{ab} commute with the Sutherland spin hamiltonian (2), this model has the Yangian symmetry Y(su(n)).

The Yangian Deformed W_∞ Algebra.

To combine the loop algebra J_n^{ab} and the Yangian algebra Q_n^{ab}, we introduce another set of generators K_n^{ab}:[6]

$$K_n^{ab} = \sum_{j=1}^{N} E_j^{ab} x_j^n. \tag{16}$$

It is easy to see that the generators K_n^{ab} represent the su(n) loop algebra,

$$[K_n^{ab}, K_m^{cd}] = \delta^{bc} K_{n+m}^{ad} - \delta^{da} K_{n+m}^{cb}, \tag{17}$$

All the K_n^{ab} can be defined recursively from the two lowest generators,

$$\begin{aligned} K_0^{ab} &= J_0^{ab}, & (18)\\ K_1^{ab} &= \sum_j E_j^{ab} x_j. & (19)\end{aligned}$$

By construction, they satisfy the relations (6-8) with J_n^{ab} replaced by K_n^{ab}.

Consider now the algebra generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$. The Yangian current Q_1^{ab} appears from an inter-relation formula between these operators;

$$[J_1^{ab}, K_1^{cd}] + [K_1^{ab}, J_1^{cd}] = 2\Big(\delta^{bc}\, Q_1^{ad} - \delta^{da}\, Q_1^{cb}\Big). \tag{20}$$

Besides the relation (15) for Q_1^{ab}, we also have the following Serre-like relations,

$$[J_0^{ab}, [J_1^{cd} + Q_1^{cd}, J_1^{ef} + Q_1^{ef}]] - [J_1^{ab} + Q_1^{ab}, [J_0^{cd}, J_1^{ef} + Q_1^{ef}]] = 0, \tag{21}$$
$$[J_0^{ab}, [K_1^{cd} + Q_1^{cd}, K_1^{ef} + Q_1^{ef}]] - [K_1^{ab} + Q_1^{ab}, [J_0^{cd}, K_1^{ef} + Q_1^{ef}]] = 0. \tag{22}$$

The relations (6-8), (14-15) and (20-22) possess an interesting interpretation: consider the generators $Q_1^{ab}(x,y)$ defined by

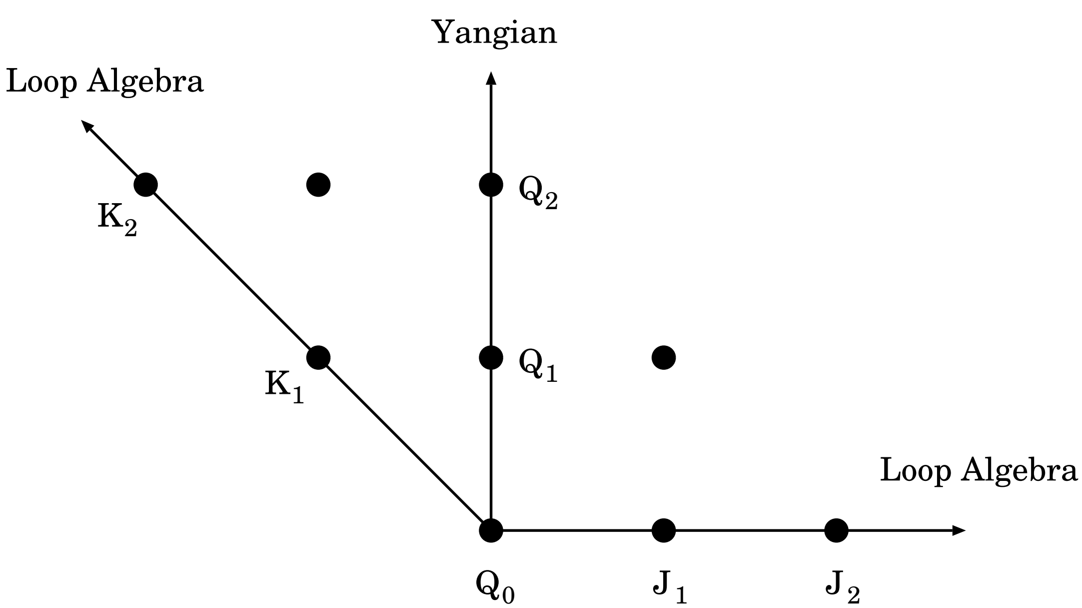
$$Q_1^{ab}(x,y) \equiv Q_1^{ab} + x\, J_1^{ab} + y\, K_1^{ab}, \tag{23}$$

for any complex numbers x and y. Then, all the previously written Serre relations can be summarized into the following compact equations :

$$[J_0^{ab}, Q_1^{cd}(x,y)] = \delta^{bc}\, Q_1^{ad}(x,y) - \delta^{da}\, Q_1^{cb}(x,y), \tag{24}$$
$$\begin{aligned} &[J_0^{ab}, [Q_1^{cd}(x,y), Q_1^{ef}(x,y)]] - [Q_1^{ab}(x,y), [J_0^{cd}, Q_1^{ef}(x,y)]] \\ &= \frac{\lambda^2}{4}\Big([J_0^{ab}, [(J_0J_0)^{cd}, (J_0J_0)^{ef}]] - [(J_0J_0)^{ab}, [J_0^{cd}, (J_0J_0)^{ef}]]\Big). \end{aligned} \tag{25}$$

In other words, the commutation relations between the generators J_n^{ab} and K_n^{ab} of the two loop subalgebras are such that the generators $Q_1^{ab}(x,y)$ form a representation of the Yangian for any x and y. We thus have an infinite number of Yangian subalgebras constructed from $Q_1^{ab}(x,y)$, but they all have λ as deformation parameter.

The algebra generated by $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$ is schematically drawn in the figure.



It is appropriate here to relate the notations in this note with those in Ref. 6; the horizontal and vertical axes indicate the node n and the spin s respectively. Corresponding to each solid circle, there exists an generator $Q_n^{(s)ab}$. The infinite symmetry associated to the operators $Q_n^{(s)ab}$ was named the quantum W_∞ algebra.[1,6]

In the limit $\lambda \to 0$, the generators J_n^{ab} reduce to $J_n^{ab} = \sum_j E_j^{ab} (\partial_{x_j})^n$. Together with the operators K_n^{ab}, they generate a W_∞-algebra with elements,

$$Q_n^{(s)ab} = \sum_{j=1}^{N} E_j^{ab} x_j^{s-1} (\partial_{x_j})^{n+s-1}, \tag{26}$$

which satisfy the commutation relations,

$$\begin{aligned} [Q_n^{(s)ab}, Q_m^{(s')cd}] \;\; = \;\; & \delta^{bc} \cdot \sum_{k=0}^{n+s-1} \frac{(n+s-1)!(s'-1)!}{k!(n+s-k-1)!(s'-k-1)!} \, Q_{n+m}^{(s+s'-1-k)ad} \\ & -\delta^{da} \cdot \sum_{k=0}^{m+s'-1} \frac{(m+s'-1)!(s-1)!}{k!(m+s'-k-1)!(s-k-1)!} \, Q_{n+m}^{(s+s'-1-k)cb}. \end{aligned} \tag{27}$$

As a consequence, this algebra is generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$. Moreover, it is easy to see that this W_∞-algebra possesses an infinite number of su(n) loop subalgebras.

For $\lambda \neq 0$, our algebra is naturally called a "Yangian deformed W_∞-algebra", and denoted $YW_\infty(su(n))$. The algebra includes the loop algebra, the Virasoro algebra,[6] and the Yangian as the subalgebras.

The Yangian Subalgebras.

We now analyze a little more the structure of the algebra. Let us first identify another Yangian subalgebra. Define another set of operators $\widetilde{Q}_1^{ab}(h,\omega)$ by

$$\widetilde{Q}_1^{ab}(h,\omega) = h^2 J_2^{ab} - \omega^2 K_2^{ab}, \tag{28}$$

where h and ω are arbitrary complex numbers. By direct computation, we see that the operators $\widetilde{Q}_1^{ab}(h,\omega)$ constitute a representation of the Yangian since they satisfy the following relations:

$$[J_0^{ab}, \widetilde{Q}_1^{cd}(h,\omega)] = \delta^{bc}\, \widetilde{Q}_1^{ad}(h,\omega) - \delta^{da}\, \widetilde{Q}_1^{cb}(h,\omega), \tag{29}$$

$$\begin{aligned} & [J_0^{ab}, [\widetilde{Q}_1^{cd}(h,\omega), \widetilde{Q}_1^{ef}(h,\omega)]] - [\widetilde{Q}_1^{ab}(h,\omega), [J_0^{cd}, \widetilde{Q}_1^{ef}(h,\omega)]] \\ & \quad = (\lambda h \omega)^2 \Big([J_0^{ab}, [(J_0 J_0)^{cd}, (J_0 J_0)^{ef}]] - [(J_0 J_0)^{ab}, [J_0^{cd}, (J_0 J_0)^{ef}]] \Big). \end{aligned} \tag{30}$$

Notice that the deformation parameter is now $2\lambda h\omega$. These Yangian generators are conserved operators for the Calogero spin model confined in a harmonic potential with hamiltonian,

$$\mathcal{H}_{CM} = h^2\, \mathcal{H}_C + \omega^2 \sum_j x_j^2. \tag{31}$$

Hence, the Calogero model (1) confined in the harmonic potential also possesses the Yangian symmetry.[9] The limit of $\omega \to 0$ corresponds to the Calogero spin model (1); in this case the Yangian symmetry reduces to the loop algebra.

This subalgebra is actually a simple example of a more general structure. As we now explain, in the Yangian deformed W_∞-algebra generated by $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$, there exists an infinite number of "slices" in which a Yangian subalgebra can be constructed.

To prove it, we need to introduce the Dunkl operators D_i for the Calogero model.[10]

$$D_i = \frac{\partial}{\partial x_i} - \lambda \sum_{j: j\neq i} \frac{1}{x_i - x_j} K_{ij}. \tag{32}$$

where K_{ij} is the operator permuting the coordinates x_i and x_j: $x_i\, K_{ij} = K_{ij}\, x_j$. We have the commutation relations:

$$D_i\, K_{ij} = K_{ij}\, D_j, \tag{33}$$

$$[D_i, D_j] = [x_i, x_j] = 0, \tag{34}$$

$$[D_i, x_j] = \delta_{ij}\,(1 + \lambda \sum_{l: l\neq i} K_{il}) - (1 - \delta_{ij})\,\lambda\, K_{ij}. \tag{35}$$

Introduce now the operators Δ_i defined by:

$$\Delta_i = (hD_i + \omega x_i + y)\,(h' D_i + \omega' x_i + y'). \tag{36}$$

They depend on the c-numbers h, ω, y and h', ω', y'. They satisfy

$$[\Delta_i, \Delta_j] = \lambda\,(h\omega' - h'\omega)\,(\Delta_i - \Delta_j)\, K_{ij}. \tag{37}$$

This relation allows us to construct a representation of the Yangian algebra. Following Ref. 5, we introduce a monodromy matrix $T(u)$ by

$$T^{ab}(u) = \delta^{ab} + \lambda\,(h\omega' - h'\omega) \sum_i \pi\Big(\frac{1}{u - \Delta_i}\Big)\, E_i^{ab}, \tag{38}$$

where π is the projection consisting in replacing K_{ij} by P_{ij} once the permutation K_{ij} has been moved to the right of the expression. The matrix $T^{ab}(u)$ satisfy,

$$[T^{ab}(u), T^{cd}(v)] = \frac{\lambda(h\omega' - h'\omega)}{u - v} \Big(T^{cb}(u)\, T^{ad}(v) - T^{cb}(v)\, T^{ad}(u)\Big) \tag{39}$$

This is another presentation of the Yangian. Therefore, the matrix (38) forms a representation of the Yangian. As usual, the quantum determinant of $T(u)$ defines a generating function of commuting operators which all commute with the matrix $T(v)$ itself.

We thus have identified an infinite number of Yangian subalgebra in the deformed W_∞-algebra. They are parametrized by the complex number h, ω, y and h', ω', y'.

Notice that their deformation parameters are $\lambda(h'\omega - h\omega')$. The previously discussed loop and Yangian subalgebras can be recovered as particular cases of this construction.

Concluding Remarks.

We would like to conclude with a few comments. In this note, we essentially worked with a specific class of representations of the algebra. But the algebra can be defined abstractly as the associative algebra generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$ with the appropriate Serre relations. So it is important to decipher the statements which are representation dependent from those which are true in the algebra. Also we did not discuss the Hopf algebra structure, if any, of our algebra. We hope that this short note not only clarifies the mathematical structures underlying the relations between the infinite symmetry[6] and the Yangian symmetry[5,7] of the Calogero-Sutherland system, but also suggests various extensions of the theories.

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